Th 1. Let $\emptyset \neq D \subseteq \mathbb{R}$, $x_0 \in \mathbb{R}$. Then :

(i) x_0 is a cluster pt (w.r.t. D) (also called non-isolated pt or accumulation pt, in notation $x_0 \in D^c$ or $x_0 \in D^a$) : $\forall \delta > 0, \exists x \in D \setminus \{x_0\}$ s.t. $|x - x_0| < \delta$, that is

$$V_\delta(x_0) \cap (D \setminus \{x_0\}) \neq \emptyset \quad \forall \delta > 0$$

(ii) $\exists$ a seq (x_n) in $D \setminus \{x_0\}$ s.t. $\lim_n x_n = x_0$

(iii) $\operatorname{dist}(x_0, D \setminus \{x_0\}) = 0$ where

$$\operatorname{dist}(x_0, D \setminus \{x_0\}) \doteq \inf \{ |x_0 - x| : x \in D \setminus \{x_0\} \}$$

From now on (unless explicitly stated otherwise), let $f : D \to \mathbb{R}$ and $x_0 \in D^c$, $\ell \in \mathbb{R}$.

Say that $f(x) \to \ell$ as $x \to x_0$ if $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t.

$$|f(x) - \ell| < \varepsilon \text{ whenever } x \in (D \setminus \{x_0\}) \cap V_\delta(x_0)$$

Th 2 (Uniqueness) Suppose also that $f(x) \to \ell'$ (in addition to $f(x) \to \ell$) as $x \to x_0$. Then $\ell = \ell'$ (crucial that $x_0 \in D^c$) (so legitimate notation : $\lim_{x \to x_0} f(x) = \ell$. Also, if $f = f'$ on $V_{\delta_0}(x_0)$ for some $\delta_0 > 0$ then $f(x) \to \ell$ iff $f'(x) \to \ell$ (as $x \to x_0$).

Th3 (Local-Boundedness Th). Suppose $\lim_{x \to x_0} f(x) = \ell$

Then $\exists$ $M > 0$ and $\delta > 0$ such that

$$(*) \quad |f(x)| \leq M \quad \forall\, x \in (D \setminus \{x_0\}) \cap V_\delta(x_0)$$

Proof. Let $\varepsilon := 1$. Then $\exists\, \delta > 0$ s.t.

$$|f(x) - \ell| < 1 \quad \forall\, x \in (D \setminus \{x_0\}) \cap V_\delta(x_0).$$

Let $M = |\ell| + 1$. Then $(*)$ holds (why?).

Note. Readjust M if necessary one can replace $(*)$ by

$$(**) \quad |f(x)| \leq M \quad \forall\, x \in D \cap V_\delta(x_0).$$

(separately consider the case when $x_0 \in D$, and otherwise).

Th 4. (Order-Preserving). Let $f : D \to \mathbb{R}$, $x_0 \in D^c$. Suppose $\alpha, \beta \in \mathbb{R}$ ~~and~~ and $\ell \in \mathbb{R}$ s.t.

$$\alpha < \lim_{x \to x_0} f(x) = \ell < \beta$$

Then $\exists\, \delta > 0$ s.t.

$$(\#) \quad \alpha < f(x) < \beta \quad \forall\, x \in (D \setminus \{x_0\}) \cap V_\delta(x_0)$$

Proof. Pick any $\varepsilon > 0$ such that $\varepsilon \leq \min\{\beta - l, l - \alpha\}$.
Then $\exists\, \delta > 0$ such that

$$|f(x) - l| < \varepsilon \quad \forall x \in (D \setminus \{x_0\}) \cap V_\delta(x_0).$$

Noting $l - \varepsilon \geq l - (l - \alpha) = \alpha$ and
$l + \varepsilon \leq l + (\beta - l) = \beta$

and

$$l - \varepsilon < f(x) < l + \varepsilon \quad \forall x \in (D \setminus \{x_0\}) \cap V_\delta(x_0)$$

we see that (#) holds.

Remark. Please state and prove the corresponding results for $\alpha = -\infty$ or $\beta = +\infty$

Cor. Suppose $f(x) \geq \beta \ \forall x \in (D \setminus \{x_0\}) \cap V_{\delta_0}(x_0)$ with some $\delta_0 > 0$. Then $\lim_{x \to x_0} f(x) \geq \beta$ provided that the limit exists (in $\mathbb{R}$).

Proof (contrapositively) Suppose not:
$l = \lim_{x \to x_0} f(x) < \beta$. Then ...
(remember $x_0 \in D^c$).

Th 5 (Local-Boundedness with non-zero limits)
Suppose $\lim_{x \to x_0} f(x) = l \neq 0$. Then $\exists\, \delta > 0$ s.t.

$$(\star) \quad \frac{|l|}{2} < |f(x)| < \frac{3|l|}{2} \quad \forall x \in (D \setminus \{x_0\}) \cap V_\delta(x_0).$$

Proof. Let $\varepsilon := \frac{|l|}{2}$ (positive!). Then $\exists\, \delta > 0$ such that

$$|f(x) - l| < \frac{|l|}{2} \quad \forall x \in (D \setminus \{x_0\}) \cap V_\delta(x_0)$$

Since $\pm(|f(x)| - |l|) \leq |f(x) - l|$ it follows that

$$|f(x)| - |l|,\ |l| - |f(x)| < \frac{|l|}{2} \quad \forall x \in (D \setminus \{x_0\}) \cap (V_\delta(x_0))$$

i.e. $(\star)$ holds.

Th 6 (Order-Preserving & Squeeze Principle).
Let $f_i : D \to \mathbb{R}$, $x_0 \in D^c$. Then

(i) Suppose $\exists\, \delta_0 > 0$ s.t. $f_1(x) \leq f_2(x)$ $\forall x \in (D \setminus \{x_0\}) \cap V_{\delta_0}(x_0)$.
Then $\lim_{x \to x_0} f_1(x) \leq \lim_{x \to x_0} f_2(x)$ provided that both exist.

(ii) Suppose $f_1(x) \leq f(x) \leq f_2(x)$ $\forall x \in D$ and that
$\lim_{x \to x_0} f_1(x) = \lim_{x \to x_0} f_2(x)$ $(= l$, say$)$ in $\mathbb{R}$.
Then $\lim_{x \to x_0} f(x)$ also exists and equals l.

Warning (ii) does <u>not</u> follow from (i)

Computation Rules. Let $f, f_1, f_2 : D \to \mathbb{R}$ and $x_0 \in D^c$

(i) $\left|\lim\limits_{x\to x_0} f(x)\right| = \lim\limits_{x\to x_0} |f(x)|$

if $\lim\limits_{x\to x_0} f(x) = l$ exists (in $\mathbb{R}$)

(ii) $k \cdot \lim\limits_{x\to x_0} f(x) = \lim\limits_{x\to x_0}\big(kf(x)\big) \quad \forall k \in \mathbb{R}$

(under the same condition as (i))

(iii) Suppose $\lim\limits_{x\to x_0} f_i(x) = l_i \quad (i = 1, 2)$. Then

$$\lim_{x\to x_0}\big(f_1(x) \pm f_2(x)\big) = l_1 \pm l_2$$

$$\lim_{x\to x_0}\big(f_1(x)\cdot f_2(x)\big) = l_1 l_2 \qquad \left(\text{in particular } \lim_{x\to x_0}\big(f_1(x)\big)^2 = l_1^2\right)$$

and $\lim\limits_{x\to x_0} \dfrac{f_1(x)}{f_2(x)} = \dfrac{l_1}{l_2}$ provided that $l_2 \neq 0$ and $f_2(x) \neq 0 \ \forall x \in D$

(iv) Suppose $\lim\limits_{x\to x_0} f(x) = l$ and $f(x) \geq 0 \ \forall x \in D$

Then $\lim\limits_{x\to x_0} \sqrt{f(x)} = \sqrt{l}$ (and $l \geq 0$)

All these follow from the corresponding results of Ch. 3 (Sequential Limits) ~~together~~ with the following sequential criterion for limits:

Th (Sequential Criterion). Let $f: D \to \mathbb{R}$, $x_0 \in D^c$ and $l \in \mathbb{R}$. Then $\Leftrightarrow$:

(i) $\lim_{x \to x_0} f(x) = l$

(ii) $\lim_n f(x_n) = l$ for all seq (x_n) in $D \setminus \{x_0\}$ convergent to x_0.

Th* Let $f: D \to \mathbb{R}$ and $x_0 \in D^c$. Then $\Leftrightarrow$:

(i) $\lim_{x \to x_0} f(x)$ exists in $\mathbb{R}$

(ii)* $\lim_n f(x_n)$ exists in $\mathbb{R}$ whenever (x_n) is a seq in $D \setminus \{x_0\}$ convergent to x_0.

But, we would like to prove the computation rule results direct from definition rather than by results of the preceding chapter.

Example. You have done the following question: if $\lim_{n\to\infty} z_n = 2$ then

$$\lim_n \frac{z_n^3-3}{z_n^2-3} = 5.$$

Hence, with $f(x) := \dfrac{x^3-3}{x^2-3}$, $D=\mathbb{R}$ and $x_0 = 2$ together with Th 6 (on sequential criterion) we have

$$\lim_{x\to x_0} \frac{x^3-3}{x^2-3} = 5 \qquad (\text{with } x_0 = 2)$$

Another method is to apply the quotient rule for function limits. And yet another important way is via definition! That is, $\forall \varepsilon>0$, to find $\delta>0$ such that

$$\left|\frac{x^3-3}{x^2-3} - 5\right| < \varepsilon \quad \forall\ x\in(\mathbb{R}\setminus\{2\})\cap V_\delta(2)$$

Note that

$$\left|\frac{x^3-3}{x^2-3}-5\right| = \frac{|x^3-5x^2+12|}{|x^2-3|} = \frac{|x-2|\,|x^2-3x-6|}{|x^2-3|}$$

and so wish to find $m, M>0$ such that

$$(\#)\ \begin{cases} |x^2-3x-6| \le M & \forall\ x\in V_\delta(2)\setminus\{2\} \subseteq (2-\delta, 2+\delta) \\ m \le |x^2-3| & \forall\ x\in V_\delta(2)\setminus\{2\} \subseteq (2-\delta, 2+\delta) \end{cases}$$

Further $x\in(2-\delta, 2+\delta)$ means $2-\delta < x < 2+\delta$

and so $4-4\delta < 4-4\delta+\delta^2 = (2-\delta)^2 < x^2$ (provided that $\delta<2$)

which implies $1-4\delta < x^2-3$

Therefore m in (#) can be taken to be $1-4\delta$ provided that $1-4\delta$ is positive (e.g. if $\delta \leq \frac{1}{8}$ then take $m = 1/2$)

With that kind of δ, M in (#) can be found accordingly: ($0 < x < 2+\delta < 3$)

$$|x^2-3x-6| \leq |x|^2+3|x|+6 < (2+\delta)^2+3\times 3+6$$
$$\leq 3^2+3\times 3+6 = 24$$

Therefore, if $x \in V_\delta(2)$ with $\delta \in (0, \frac{1}{8}]$, one has

$$\left|\frac{x^3-3}{x^2-2} - 5\right| \leq \frac{24\,|x-2|}{1/2} = 48|x-2| < 48\delta \leq \varepsilon$$

provided that my $\delta > 0$ satisfies the additional requirement that $\delta \leq \varepsilon/48$ (in addition to $\delta \leq \frac{1}{8}$). Therefore the formal proof can be as follows:

Let $\varepsilon > 0$. Take $\delta := \min\left\{\frac{1}{8}, \frac{\varepsilon}{48}\right\}$ (so δ is a positive number & $\delta \leq \frac{1}{8}$, $\delta \leq \frac{\varepsilon}{48}$)

Suppose $|x-2| < \delta$. Then

$$\left|\frac{x^3-3}{x^2-2} - 5\right| = \frac{|x-2|\,|x^2-3x-6|}{|x^2-3|} \leq \frac{|x-2|\cdot 24}{1/2} < 48\delta \leq \varepsilon$$

because $|x-2| < \delta \leq \frac{1}{8}$ so $\frac{15}{8} < x < 2+\frac{1}{8} < 3$

and $\frac{1}{2} < x^2-3$ and $|x^2-3x-6| < 3^2+9+6 = 24$

($\frac{1}{2}$ and 24 are not the "best" but they serve the job!)